International Conference on Machine Learning (ICML) 2018

On the Optimization of Deep Networks: Implicit Acceleration by Overparameterization

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How Does Depth Help?

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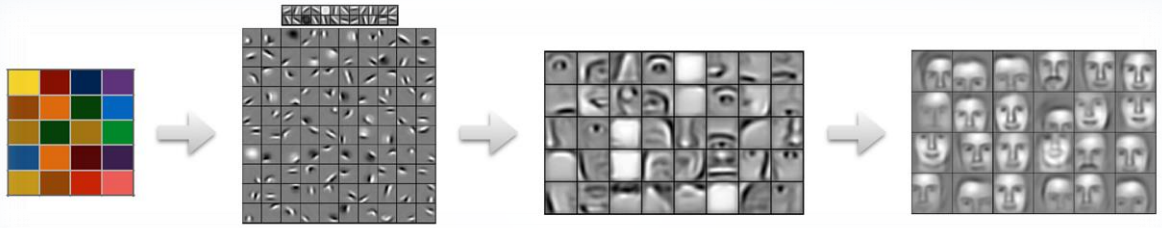
Conventional wisdom:

- Depth boosts expressive power



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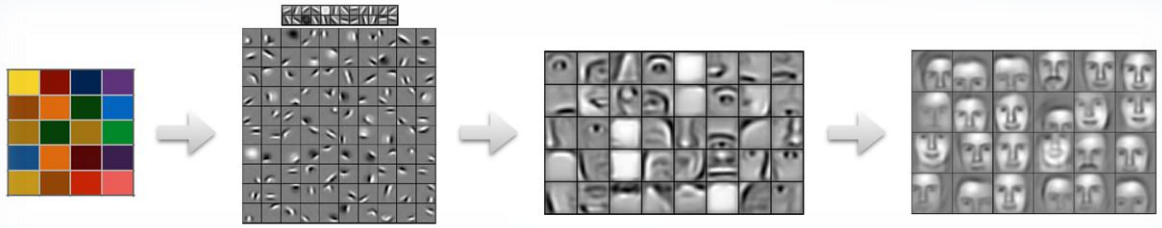

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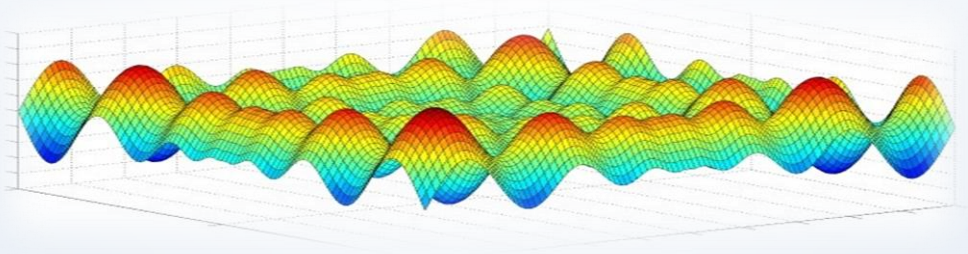

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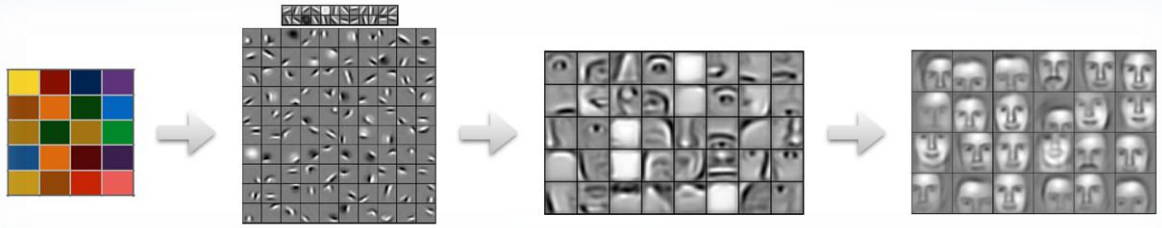

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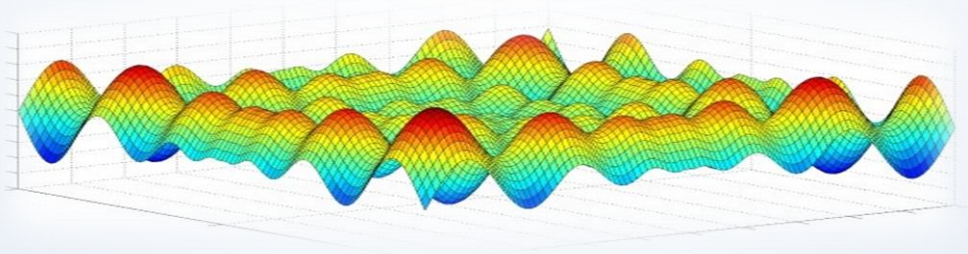

This work:

Depth can accelerate optimization!

Decoupling Optimization from Expressiveness

Problem:

Expressiveness can interfere with our study – deeper nets may seem to optimize faster per being able to reach lower training err

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- Adding layers amounts to replacing matrix param by product of matrices – **overparameterization**

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Linear neural networks were studied extensively, cf. [Saxe et al. 2013; Kawaguchi 2016; Hardt & Ma 2016]

Warm-Up

Training objective for linear regression with ℓ_p loss:

$$L(\boldsymbol{w}) = \sum_{(\boldsymbol{x},y)\in S} \frac{1}{p}\left(\boldsymbol{x}^T\boldsymbol{w} - y\right)^{\boldsymbol{p}}$$

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Overparameterize – replace vector $\boldsymbol{w}$ by vector $\boldsymbol{w_1}$ times scalar w_2:

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- Gradient descent (GD) over $\tilde{L}(\boldsymbol{w_1}, w_2)$ induces on $\boldsymbol{w} = \boldsymbol{w_1}w_2$:

$$\boldsymbol{w}^{(t+1)} \hookleftarrow \boldsymbol{w}^{(t)} - \rho^{(t)} \cdot \nabla L\left(\boldsymbol{w}^{(t)}\right) - \sum_{\tau=1}^{t-1} \mu^{(t,\tau)} \cdot \nabla L\left(\boldsymbol{w}^{(\tau)}\right)$$

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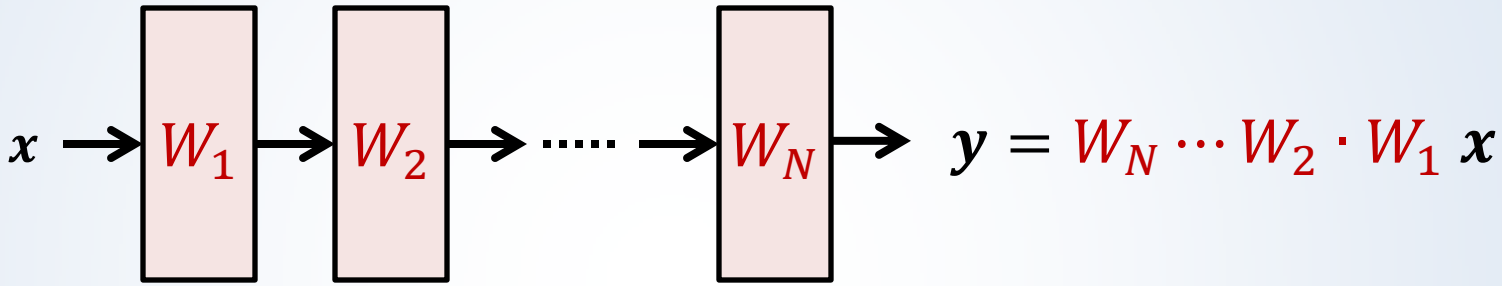
Formal Setup

Linear neural network:



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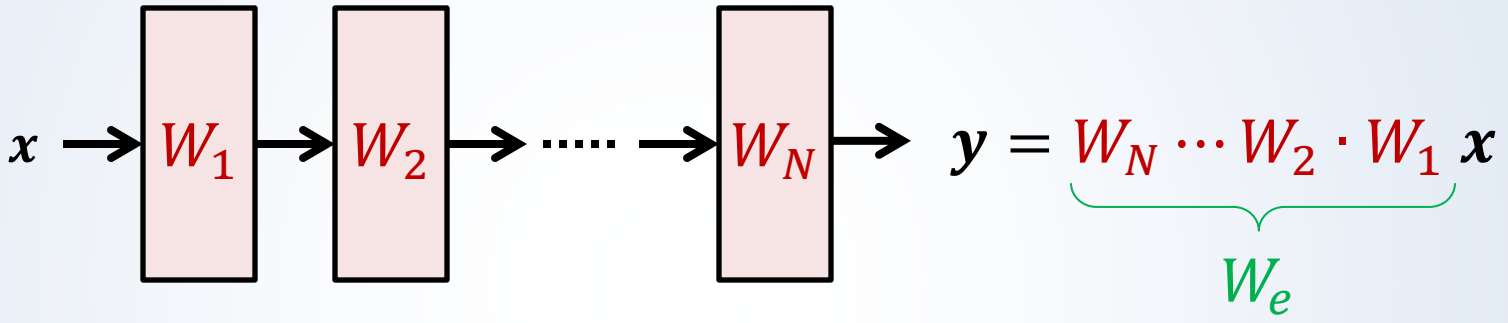

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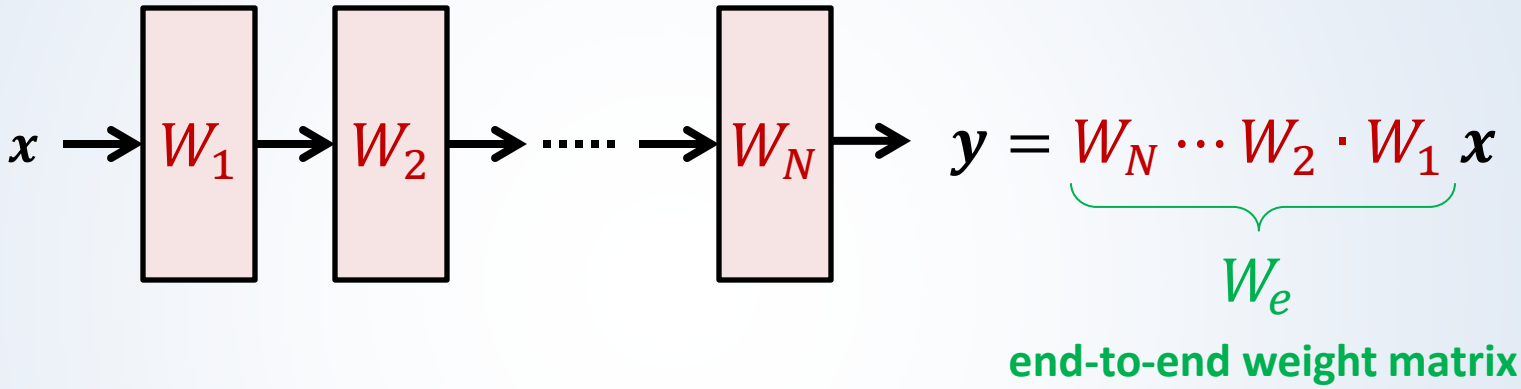

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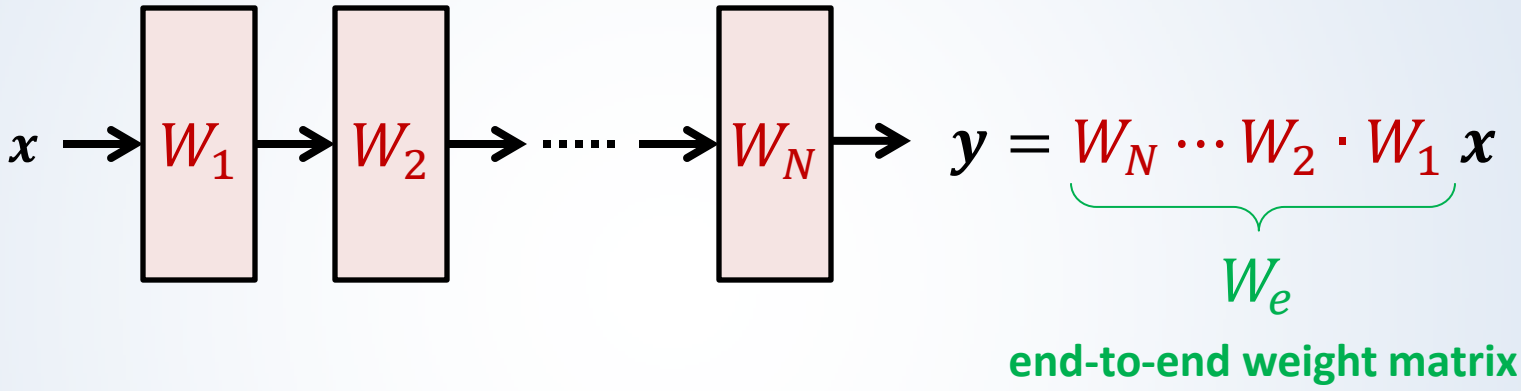

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Question:

How does W_e behave during GD over $W_1, \cdots, W_N$?

Implicit Dynamics of Gradient Descent

Theorem:

GD over $W_1, \cdots, W_N$ with small learning rate and near-zero init, induces on W_e the **end-to-end update rule**:

$$W_e^{(t+1)} \longleftarrow W_e^{(t)} - \eta \sum_{j=1}^{N} \left[W_e^{(t)} \left(W_e^{(t)} \right)^T \right]^{\frac{j-1}{N}} \nabla L\left(W_e^{(t)} \right) \left[\left(W_e^{(t)} \right)^T W_e^{(t)} \right]^{\frac{N-j}{N}}$$

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Overparameterization with deep linear net induces on GD a certain acceleration scheme! No dependence on layer widths!

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Fundamental theorem for line integrals:

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Effect of overparameterization by depth cannot be attained via any modification of the objective!

Experiments – Linear Networks

Regression problem from UCI ML Repository; ℓ_4 loss

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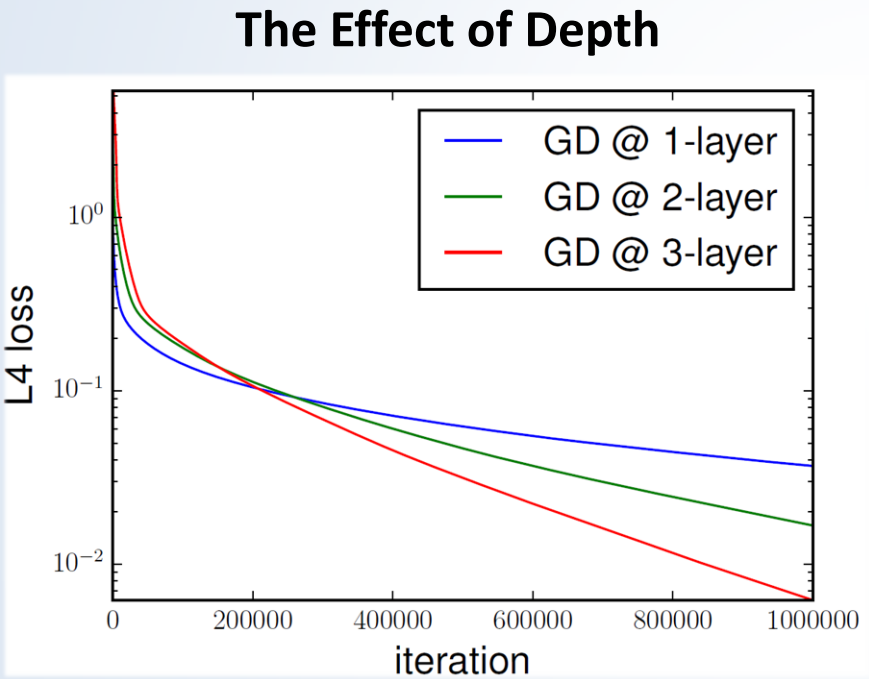

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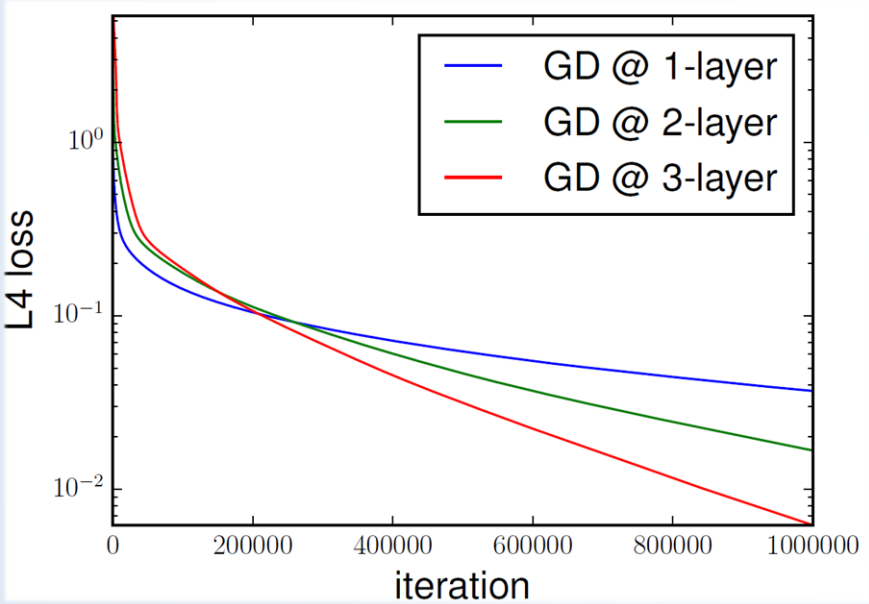

Depth can speed up GD, even w/o any change in expressiveness, and despite introducing non-convexity!

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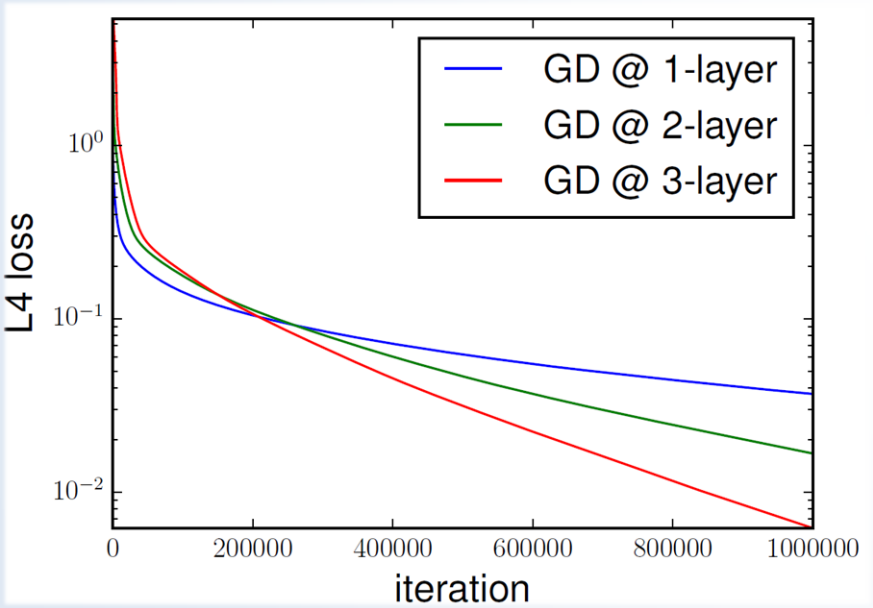

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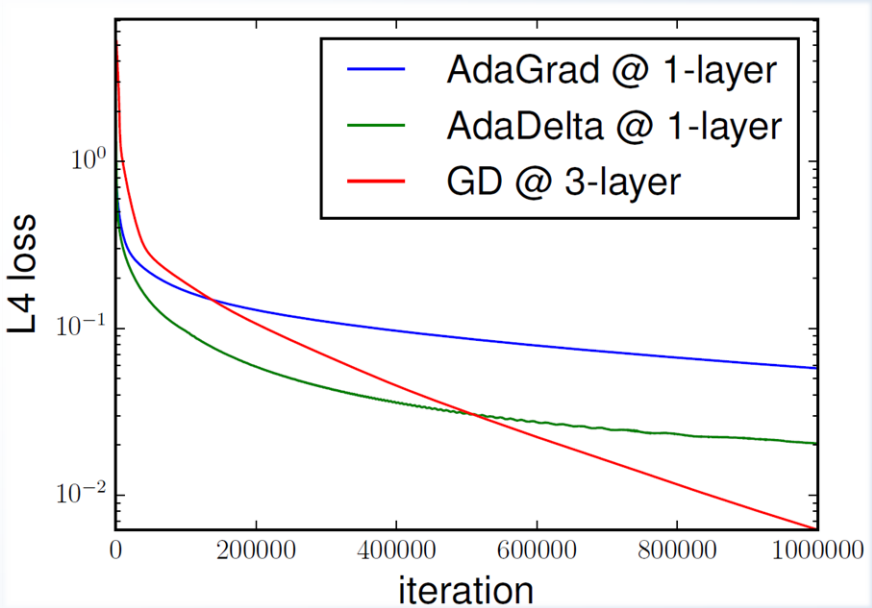

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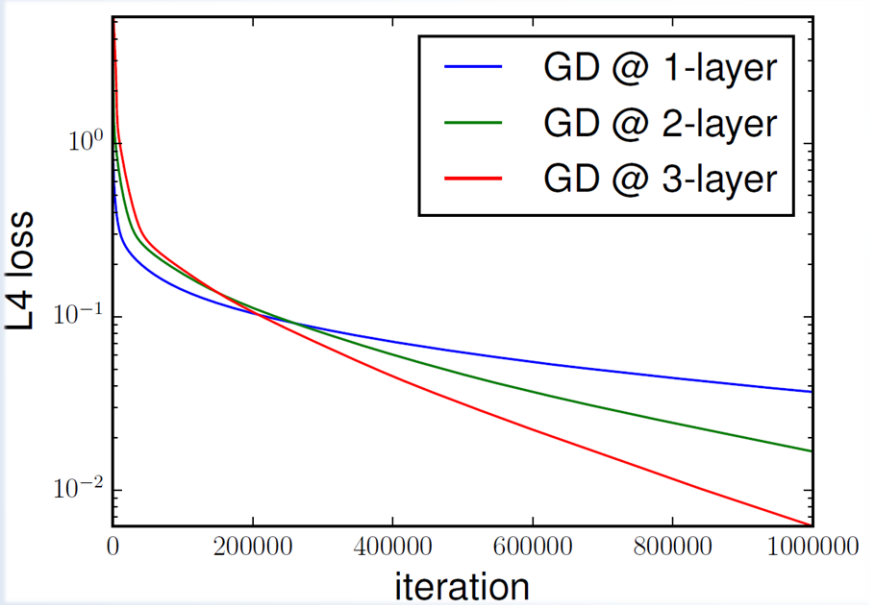

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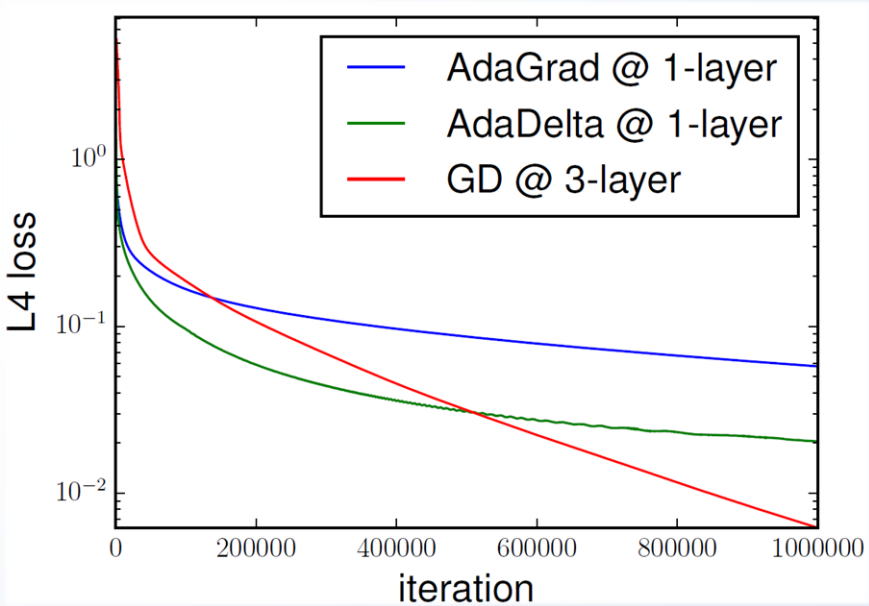

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This speed up can outperform explicit acceleration methods designed for convex problems!

Experiments – Non-Linear Network

TensorFlow CNN tutorial for MNIST

Overparameterization: fully-connected layers → depth-2 linear nets

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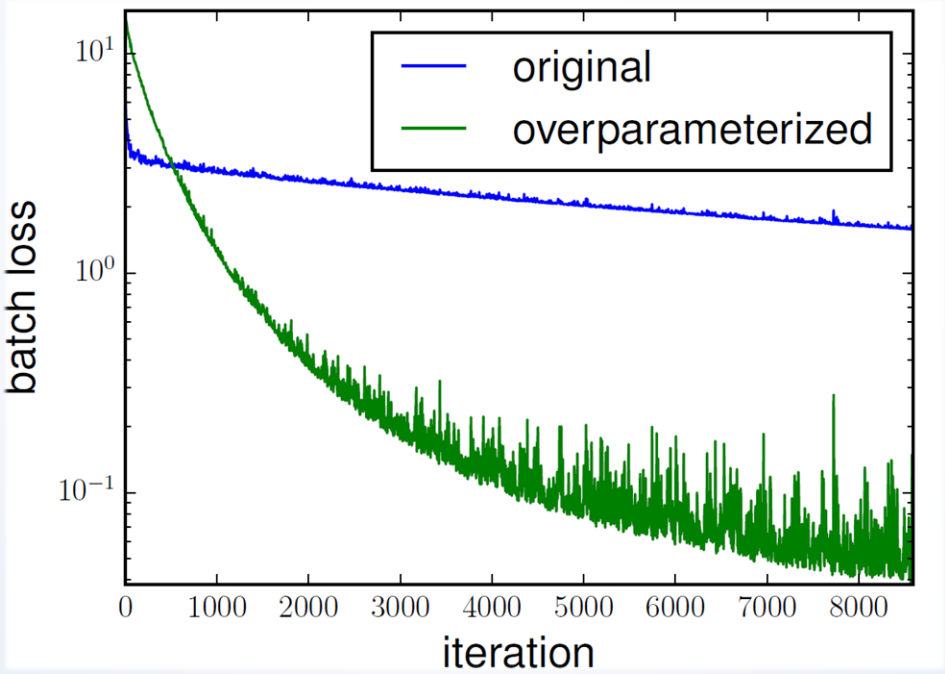

With +15% in params, and no change in expressiveness, overparameterization accelerated by orders of magnitude!

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- We show:
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  - Preconditioning combines adaptive learning rate and "momentum"
  - Effect cannot be attained via any modification of original objective
  - Can lead to significant speed ups, despite no change in expressiveness

Thank You